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# MATHEMATICAL LIFE

## On the 90th anniversary of the birth of Vladimir Nikolaevich Sudakov (1934–2016)

Vladimir Nikolaevich Sudakov was born on 10 January 1934 in Leningrad, where he also spent his childhood. During the Leningrad siege he remained in the city, continuing school studies. When asked about his attitude to lessons of German language at a school to which he was making his way under Nazi bombs, Sudakov answered: "There was a banner "Deutsche Sprache ist eine Waffe" at the entrance to our school. This was my attitude." His grandfather, who shared his bread ration with the grandson, died of exhaustion during the siege, and his father, who defended the approaches to Leningrad, was badly wounded.

After school Vladimir enrolled at the Faculty of Mathematics and Mechanics at Leningrad University, where as a student he was greatly influenced by Gleb Pavlovich Akilov. After graduating successfully from the university Sudakov was admitted for postgraduate studies to the Leningrad Department of the Steklov Mathematical Institute, where Leonid Vital'evich Kantorovich was his scientific advisor.

Subsequently, Sudakov was employed by the Laboratory of Statistical Methods of the institute, at the head of which was at that time Yurii Vladimiroivich Linnik. He worked there till the end of his life.

Sudakov's research covered a wide circle of problems in functional analysis, measure theory, probability theory, stochastic processes, and mathematical statistics. The unifying motif of his investigations were perhaps measures and compact subsets of infinite-dimensional spaces. In 60 years, starting with the publication of his first paper [21], submitted to a journal in 1956 and published in 1957, he published dozens of articles and the outstanding monograph [37].

The result of his first paper [21] is an interesting observation concerning the criteria for the compactness of sets in $L^p$ due to A. N. Kolmogorov and M. Riesz.

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Kolmogorov showed that a set $G$ in $L^p(F)$, where $F$ is a bounded measurable subset of $\mathbb{R}^n$, lies in a compact set precisely when it is bounded in norm and satisfies the condition

$$\lim_{h\to 0}\sup_{f\in G}\|f-f_h\|_{L^p}=0, \tag{1}$$

where $f_h$ is a Steklov average, that is,

$$f_h(x)=\frac{1}{|S_h(x)|}\int_{S_h(x)} f(y)\,dy,$$

where $S_h(x)$ is the ball with centre $x$ and radius $h$ and $|S_h(x)|$ is the volume of this ball. This yields a criterion for the compactness of a bounded subset $G$ of $L^p(\mathbb{R}^n)$: Kolmogorov's condition (1) must be supplemented by

$$\lim_{R\to\infty}\sup_{f\in G}\int_{x:\ |x|\geqslant R}|f(x)|^p\,dx=0. \tag{2}$$

In a close criterion due to M. Riesz, the condition

$$\lim_{h\to 0}\sup_{f\in G}\int_{\mathbb{R}^n}|f(x+h)-f(x)|^p\,dx=0 \tag{3}$$

is imposed in place of (1). It was shown in [21] that conditions (1) and (2) (or (3) and (2)) imply that $G$ is bounded in norm, so this need not be assumed in advance. Of course, it is important here that functions on the whole of $\mathbb{R}^n$, rather than on a bounded set, are under consideration. Infinite-dimensional compact sets were one of Sudakov's favourite topics, which appeared in many of his works.

His next paper [22] opened way to his second favourite topic, measures in infinite-dimensional spaces. In the 1950s integration in function spaces was an actively developing field; this was motivated by the needs of the theory of stochastic processes and quantum field theory. Some interesting problems arising in this field were of general theoretic interest, for instance, problems of infinite-dimensional analogues of Haar measure. There can be no full analogue, just as there are no Borel probability measures equivalent to their translations. However, there are measures with large stocks of quasi-invariance vectors (such that the translations by these vectors are equivalent to the measure in question). For a set $K$ in a separable Hilbert space $H$ the existence of a Borel probability measure $\mu$ such that all translations $\mu(\,\cdot-k)$ by vectors $k\in K$ are equivalent to $\mu$, is tantamount to $K$ lying in the range of some Hilbert–Schmidt operator, that is, a bounded operator $T$ such that for some (and therefore each) orthonormal basis $\{e_n\}$ the norms $\|Te_n\|^2$ form a convergent series. There is no criterion for general Banach space, nor even for the space of continuous functions $C[0,1]$. It is an easy observation that such a set must be covered by a countable family of compact sets (so that it cannot coincide with the whole infinite-dimensional space). In [22] and [25] Sudakov established more subtle results. In [22] he showed that if a finite measure is defined on a $\sigma$-algebra of subsets of a non-separable metric linear space $E$ that contains all balls of $E$, then it cannot be quasi-invariant under all translations. In [25] it was proved that a closed balanced set can consist of quasi-invariance vectors precisely when it has the form $S + L$, where $L$ is a finite-dimensional subspace and $S$ is a subset of a Hilbert–Schmidt ellipsoid. The results in these two papers made up the main content of Sudakov's Ph.D. thesis (see [24]), which he defended in 1962. Measure theory on infinite-dimensional spaces was also the subject of [23] and [43].

In the joint paper [19] with the well-known Polish mathematician A. Pelczyński they generalized a famous result of R. S. Phillips that $c_0$ cannot be complemented in $l^\infty$: they showed that in the Banach space $m(S)$ of bounded functions on an uncountable set $S$ the subspace $m_0(S)$ of functions $f$ such that for each $\varepsilon > 0$ the set $\{s\colon |f(s)| > \varepsilon\}$ has cardinality less than that of $S$ is uncomplementable.

In [26] Sudakov proposed the following interesting characterization of subsets of Hilbert–Shmidt ellipsoids. Let $H$ be a separable Hilbert space, and let $M \subset H$. Then $M$ lies in a Hilbert–Schmidt ellipsoid if and only if for each orthonormal basis $\{e_n\}$ the quantity

$$S(\{e_n\}) = \sum_{n=1}^{\infty} \sup_{m\in M} |(m, e_n)|^2$$

is finite. In this case the supremum of such quantities over all orthonormal bases is also finite.

In the joint paper with I. V. Romanovskii [20] they touched on an issue which remained the focus of Sudakov's attention till the end of his days, namely, Lebesgue–Rokhlin spaces and conditional measures. His paper [30] and his last work [39] were also devoted to this subject.

Sudakov's communication [27] at the International Congress of Mathematicians in Moscow (1966) opened a new line of research, which brought him international recognition. It was inspired by a question of Kolmogorov on conditions for the continuity of trajectories of a Gaussian process in terms of its covariation function. In Sudakov's approach the geometry of subsets of a Hilbert space connected with the process is of importance. Namely, the properties of a Gaussian process $(X(t))_{t\in T}$ defined on an arbitrary parameter set $T$ are studied with the help of the half-metric

$$\varrho(s,t) = [\mathsf{E}(X(s) - X(t))^2]^{1/2}.$$

If $T$ turns out to be separable in this metric, then there exists a separable modification of the process $X$ on the same probability space. Another outstanding result due to Sudakov connects the first intrinsic volume $V_1$ of a convex body in a Hilbert space with the maximum of an isonormal Gaussian process on this set. In Proposition 14 in [37] he proved the following relation: for each convex compact set $T \subset H$,

$$V_1(T) = \sqrt{2\pi}\,\mathsf{E}\sup_{t\in T} X(t).$$

It should be noted that the celebrated Fernique–Sudakov inequality was derived by Sudakov directly from this representation and the purely geometric result stating that if, after some ordering, the lengths of edges of a multidimensional simplex do not exceed the lengths of edges of another simplex, then this inequality also holds for their first volumes (see [37], Theorem 2).

The main results obtained in this area were presented in the proceedings of conferences [28] and [35] and in the short notes [29] and [31], and then they were summarized in the monograph [37]. This monograph, which was a remarkable contribution to the theory of infinite-dimensional probability distributions and general measure theory grew out of Sudakov's D.Sc. thesis [33], [34], which he defended in 1973. At approximately the same time close results were independently obtained by R. M. Dudley from the USA and the French mathematician X. Fernique. Although a complete solution to the problem of characterization of a continuous Gaussian process was only found much later by M. Talagrand, the Dudley–Fernique–Sudakov conditions proved to be most useful ones in practice.

Sudakov's another outstanding contribution to the theory of Gaussian measures was made in the joint papers [36] and [13]. In the first the authors established the famous Sudakov–Tsirelson–Borell inequality (Christer Borell proved it independently at approximately the same time): if $V$ is a convex measurable set in a space with centred Radon Gaussian measure $\gamma$ and $\Pi$ is a measurable half-space of the form $\{l \geqslant 0\}$ for some measurable linear functional $l$, and if $\gamma(V) \geqslant \gamma(\Pi)$, then

$$\gamma(sV) \geqslant \gamma(s\Pi) \quad \text{for all} \quad s \geqslant 1.$$

Another important result in [36] was the inequality

$$\Phi^{-1}(\gamma(A+\delta U)) \geqslant \Phi^{-1}(\gamma(A)) + \delta, \qquad \delta > 0,$$

where $\gamma$ is a countable power of the standard Gaussian measure on the line, $A$ is a Borel set of positive $\gamma$-measure, and $U$ is the closed unit ball in the Hilbert space $l^2$. This inequality also holds for each centred Radon Gaussian measure $\gamma$, provided that $U$ is the closed unit ball in the Cameron–Martin Hilbert subspace of $\gamma$ (in the case of a countable power of the standard Gaussian measure this is the space $l^2$). In [13] the authors proved quite an important result: if a Borel function $f$ is Lipschitz relative to the Cameron–Martin subpace $H$ of a measure $\gamma$, that is, $|f(x+h)-f(x)| \leqslant C|h|_H$ for $h \in H$, then $\exp(\varepsilon f^2)$ is integrable with respect to $\gamma$ for $\varepsilon < (2C^2)^{-1}$. Both papers contain also a number of important results on the distribution functions of seminorms on spaces with Gaussian measures.

The first proof of the Gaussian isoperimertic inequality was based on a difficult isoperimetric theorem on a sphere (going back to P. Lévy and E. Schmidt). Subsequently, the Gaussian case became a starting point for many investigations, new approaches and tools, applications and further generalizations from the standpoint of the 'phenomenon of measure concentration' (the term proposed by V. D. Milman in the early 1970s to point out the role of dimension for deviations of Lipschitz functions on a sphere).

The phenomenon of measure concentration on a sphere manifests iteself, for example, when we consider the distributions of linear functionals on a multidimensional Euclidean space with respect to isotropic probability measures. It was an important discovery of Sudakov's in 1978 [38] that almost all such distributions concentrate around a certain typical distribution on a line. This short note gave an impetus to the development of a wide area at the interface between functional analysis and probability theory; we refer the reader to the monograph [6], devoted to Sudakov's theorem and the description of its further development.

Apart from results on the properties of trajectories of Gaussian processes, [37] presents extensive material (Chap. III, taking more than 100 pages) relating to measurable partitions, conditional measures, bistochastic measures and operators, and the Birkhoff and Monge problems. The main result in this direction (announced in the note [32]) was the proof of the existence of an independent complement to two given measurable partitions of the standard measure space under broad assumptions. This can be regarded as an answer to a question of G. Birkhoff. In the final section, § 13, of Chap. III Sudakov proposed a solution to the Monge optimal transportation problem. This problem, goes back to G. Monge's studies of the end of the 18th century, and in its modern form (stated by A. M. Vershik in 1970) consists in the following. Given two Borel probability measures $\mu$ and $\nu$ on $\mathbb{R}^n$ with finite first moments, find a Borel transformation of $\mu$ into $\nu$ that minimizes the integral

$$\int_{\mathbb{R}^n} |T(x) - x| \, \mu(dx)$$

over all transformations of $\mu$ into $\nu$. Of course, this problem has an abstract version for general measure spaces and cost functions $h(x, y)$ in place of $|x - y|$ (where the integral of $h(T(x), x)$ with respect to $\mu$ must be minimized). However, even for $|x - y|$ the minimum is not necessarily achieved in the case of singular measures. Monge himself treated the case when both measures were the restrictions of the standard volume to two sets of equal volume. The existence of a minimum was believed to be obvious, and the properties of possible minimizing maps were under study. Only in 1970, when Vershik drew attention to this question, did it become clear that the existence of a solution to the Monge problem had not been proved. Sudakov attacked this problem and developed a method for constructing a solution in the case of an arbitrary (not necessarily Euclidean) norm on $\mathbb{R}^n$ and absolutely continuous measures with finite first moments. This method uses essentially conditional measures and measurable distributions. For more than 20 years it had been believed well established that the Monge problem had a solution under these assumptions, until some doubts arose about Sudakov's argument. The further development was rather dramatic. In [2] a counterexample was produced to the questionable auxiliary technical result of Sudakov, concerning delicate questions of the use of conditional measures. Thus, after 25 years the central result was called into question. However, several groups of authors (see [2], [3], [8], [15], [41], [11], and [9]) managed to find correct, although very lengthy and complicated, proofs of important special cases of Sudakov's theorem, including the case of the standard Euclidean norm (even this case was in doubt before that). Finally, in [12] all restrictions on the norm were lifted. Of course, Sudakov was very disturbed by the news of a gap, not to speak about the counterexample to a part of his proofs, so he was enjoyed to know at the end of his life that his theorem was fully rehabilitated. Moreover, while all proofs in the above papers used quite different approaches, in [4], [5], and [10] his approach was also exonerated. Namely, it was shown that the result can also be justified on the way taken by Sudakov, once one modifies the technical result containing a gap.

The remarkable results established by Sudakov were not only collected in his monograph [37], but were also included in books by many other authors; for instance, see [1], [6], [7], [16], [42], [17], [18], and [40]. The notes [14] and [44] present comments on Sudakov's work and memories of him.

Apart from mathematics, Sudakov had a deep knowledge in literature and history and assembled a rich collection of books on these areas. He was also avidly interested in various questions of physics, in which he consulted his friend Leonid Aleksandrovich Khalfin. With his wife Lyudmila Nikolaevna, they raised a son Andrei, who is also a mathematician.

Vladimir Nikolaevich Sudakov left a bright mark in science and in the hearts of all those who knew him. His contribution to mathematics and his broad erudition are a source of inspiration for new generations of experts and researchers.

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